

# COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS WITH WEAK LYAPUNOV INSTABILITY OR ALL PERIODIC ORBITS ON NON-COMPACT LEVEL SETS

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**ABSTRACT.** The aim of this paper is to introduce a class of Hamiltonian autonomous systems in dimension 4 which are completely integrable and their dynamics is described in all details. They have an equilibrium point which is stable in some rare elements of the class and unstable in most cases. Anyhow, the eigenvalues of the linearization at the equilibrium point are imaginary and no motion is asymptotic in the past, namely no solution has the equilibrium as limit point as time goes to minus infinity. In the unstable cases, there is a sequence of initial data which converges to the origin whose corresponding solutions are unbounded and the motion is slow. So instability is quite weak and perhaps no such explicit examples of instability are known in the literature. The stable cases are also interesting since the level sets of the 2 first integrals independent and in involution keep being non-compact and stability is related to the isochronous periodicity of all orbits and the existence of a further first integral.

## 1. INTRODUCTION

Let us have a quick look of our main results, the details and the proofs are in the paper. We introduce the Hamiltonian system in  $\mathbb{R}^4$

$$(1.1) \quad \begin{cases} \dot{q}_1 = p_2 \\ \dot{q}_2 = p_1 \\ \dot{p}_1 = -g'(q_1) q_2 \\ \dot{p}_2 = -g(q_1) \end{cases}$$

where  $g$  is a  $C^1$  function near 0 and satisfies  $g(0) = 0$  and  $g'(0) > 0$ . The Hamiltonian function for the system is

$$(1.2) \quad H(q, p) = p_1 p_2 + g(q_1) q_2$$

and another first integral is

$$(1.3) \quad K(q, p) = \frac{p_2^2}{2} + V(q_1), \quad V(q_1) = \int_0^{q_1} g(s) ds.$$

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The  $q_1, p_2$ -plane is invariant and the origin is a center on it, namely an open neighbourhood  $C$  of  $(0, 0)$  in this plane is the union of periodic orbits which enclose  $(0, 0)$ . We restrict the phase space to  $M = \{(q_1, q_2, p_1, p_2) \in \mathbb{R}^4 : (q_1, p_2) \in C, (q_2, p_1) \in \mathbb{R}^2\}$ , so we have a global center in the  $q_1, p_2$ -plane. The Hamiltonian vector fields  $\Omega \nabla H(q, p)$  in (1.1) and  $\Omega \nabla K(q, p)$  are both complete on  $M$  (with  $\Omega$  skew-symmetric matrix, see (4.4)). The functions  $H, K$  are in involution. The vectors  $\nabla H(q, p), \nabla K(q, p)$  are linearly independent at each point of the set  $N := \{(q, p) \in M : (q_1, p_2) \neq (0, 0)\}$  which is invariant for both the Hamiltonian vector fields  $\Omega \nabla H$  and  $\Omega \nabla K$ .

A nonempty component  $\Gamma \subset N$  of a level set of  $H, K$  is diffeomorphic to  $\mathbb{S}^1 \times \mathbb{R}$  and there are coordinates  $\phi \bmod 2\pi$  and  $z$  on  $\mathbb{S}^1 \times \mathbb{R}$  such that the differential equations defined by the vector field  $\Omega \nabla H$  on  $\Gamma$  take the form

$$(1.4) \quad \dot{\phi} = \omega, \quad \dot{z} = v \quad (\omega, v = \text{const}).$$

If  $(x_0, 0)$  belongs to the projection of  $\Gamma$  on the  $q_1, p_2$ -plane and  $\mathcal{T}(x_0)$  is the period of the first component of any integral curve of  $\Omega \nabla H$  on  $\Gamma$ , then  $v = 0$  if and only if  $\mathcal{T}'(x_0) = 0$ .

A necessary and sufficient condition for the stability of the origin of (1.1) is that all periodic orbits in a neighbourhood of  $(0, 0)$  in the  $q_1, p_2$ -plane have the same period so the center is (locally) isochronous. In this case all orbits of (1.1) in  $\mathbb{R}^4$ , with  $(q_1, p_2)$  near  $(0, 0)$ , are periodic and have the same period. Moreover, a further first integral appears as we shall see using a result of Barone and Cesar [3]. This happens for instance for the following two functions

$$(1.5) \quad g(x) = 1 - \frac{1}{\sqrt{1+4x}}, \quad g(x) = 1 + x - \frac{1}{(1+x)^3}.$$

In Section 2 we show a theorem on isochronous centers taken from [6]. We reproduce here the proof to be self-contained. Our result relates these functions with the even functions and so it shows that isochronicity in the  $q_1, p_2$ -plane, and Lyapunov stability for (1.1), is a rare phenomenon among the class (1.1). Isochronous centers were first studied by Urabe [5] with a different approach, in [6] a new characterization by means of the so called ‘involutions’  $h$  was found which permits to construct the functions giving isochronous centers.

Generally we have instability of the equilibrium for (1.1) and it is very easy to show explicit functions for it, for instance

$$(1.6) \quad g(x) = \sin x, \quad g(x) = \alpha x + \beta x^2 + \gamma x^3,$$

with  $\alpha > 0, (\beta, \gamma) \neq (0, 0)$ . This kind of instability is quite weak since the linearization of the system (1.1) at the origin in  $\mathbb{R}^4$  has the double eigenvalues  $\pm i\sqrt{g'(0)}$ . Moreover, there are no asymptotic motions to the equilibrium, namely no motions with the origin as limit point as

$t \rightarrow -\infty$ . In the present paper, instability without asymptotic motions is called *weak instability*.

In our Hamiltonian systems, instability occurs because there is a sequence of initial data which converges to the origin whose corresponding solutions are unbounded and the motion is slow, indeed (1.4) shows that the coordinate  $z$  is an affine function of time.

*Acknowledgement.* The picture in Section 3 was made using the application *Mathematica* by Wolfram Research Inc. by means of the package *CurvesGraphics6* by Gianluca Gorni of the University of Udine available on [www.dimi.uniud.it/~gorni](http://www.dimi.uniud.it/~gorni).

## 2. ISOCHRONOUS OSCILLATIONS

Let us start from the following equation

$$(2.1) \quad \ddot{x} = -g(x), \quad g(0) = 0, \quad g'(0) > 0,$$

where  $g$  is continuous in a neighbourhood of 0 in  $\mathbb{R}$  and differentiable at 0 (in Section 1 the function  $g$  was  $C^1$ , while in the present Section 2 the existence of  $g'(0)$  is enough). This o.d.e. has the first integral of energy

$$(2.2) \quad G(x, \dot{x}) = \frac{\dot{x}^2}{2} + V(x), \quad V(x) = \int_0^x g(s) ds.$$

By means of this first integral, we easily see that each Cauchy problem for (2.1) has a unique solution if  $g(x) = 0$  only at  $x = 0$ .

The potential energy  $V$  is a  $C^1$  function and there exists  $V''(0) = g'(0) > 0$ . We can restrict the domain of  $g$  to an open interval  $J \ni 0$  such that  $V$  is strictly increasing on  $J \cap \mathbb{R}_+$ , strictly decreasing on  $J \cap \mathbb{R}_-$ , and for each point  $x \in J$  there is a unique point  $h(x) \in J$  with

$$(2.3) \quad V(h(x)) = V(x), \quad \text{sgn}(h(x)) = -\text{sgn}(x),$$

where  $\text{sgn}(x)$  is the sign of  $x$ . We check at once that the function

$$(2.4) \quad u(x) := \text{sgn}(x) \sqrt{2V(x)}$$

is a  $C^1$  diffeomorphism onto the image  $I = u(J)$  which is a symmetric interval

$$(2.5) \quad \begin{aligned} u'(0) &= \sqrt{V''(0)}, \quad u \in \text{Diff}^1(J; I), \\ 0 \in J, \quad 0 \in I, \quad y \in I &\implies -y \in I. \end{aligned}$$

Moreover, since  $h(x) = u^{-1}(-u(x))$  we have that  $h$  is also a diffeomorphism:

$$(2.6) \quad \begin{aligned} h &\in \text{Diff}^1(J; J), \quad h(h(x)) = x, \\ h(0) &= 0, \quad h'(0) = -1. \end{aligned}$$

We call  $h$  the *involution* associated with  $V$ . Remark that the graph of  $h$  is symmetric with respect to the diagonal which intersects at the origin;

indeed  $(x, h(x))$  has  $(h(x), x)$  as symmetric point and this coincides with the point  $(h(x), h(h(x)))$  of the graph.

**Theorem 2.1** (Determining isochronous centers). *Let  $V$  be a  $C^1$  function near 0 in  $\mathbb{R}$  with  $V(0) = V'(0) = 0$ , assume there exists  $V''(0) > 0$ , let  $J$  be an open interval as above and let  $h \in \text{Diff}^1(J; J)$  be the involution associated with  $V$ . Then all orbits of  $\ddot{x} = -V'(x)$  which intersect the  $J$  interval of the  $x$ -axis in the  $x, \dot{x}$ -plane, are periodic and enclose  $(0, 0)$ . Moreover, they all have the same period if and only if*

$$(2.7) \quad V(x) = \frac{V''(0)}{8} (x - h(x))^2, \quad x \in J.$$

In this case we say that the origin is an isochronous center for  $\ddot{x} = -V'(x)$ .

*Proof.* By composition with the inverse of the diffeomorphism (2.4), the first integral (2.2) gives  $2G(u^{-1}(y), \dot{x}) = \dot{x}^2 + y^2$ . The first part of the thesis follows at once.

Now, consider a periodic orbit in the  $x, \dot{x}$ -plane which intersects the  $x$ -axis at  $x_0 \in J$ ,  $x_0 > 0$ , then  $h(x_0) < 0$  is the other intersection. By the energy conservation we get the period of the orbit as

$$(2.8) \quad \mathcal{T}(x_0) = 2 \int_{h(x_0)}^{x_0} \frac{dx}{\sqrt{2(V(x_0) - V(x))}}.$$

If  $y_0 \in I$ ,  $y_0 > 0$ ,  $x_0 = u^{-1}(y_0)$ , then  $h(x_0) = u^{-1}(-y_0)$ , and the period

$$(2.9) \quad \begin{aligned} T(y_0) &:= 2 \int_{u^{-1}(-y_0)}^{u^{-1}(y_0)} \frac{dx}{\sqrt{2(V(x_0) - V(x))}} = \\ &= 2 \int_{-y_0}^{y_0} \frac{(u^{-1})'(r) dr}{\sqrt{y_0^2 - r^2}} = 2 \int_0^{y_0} \frac{((u^{-1})'(r) + (u^{-1})'(-r)) dr}{\sqrt{y_0^2 - r^2}}. \end{aligned}$$

The change of integration variable  $s = \arcsin(r/y_0)$  gives for  $y_0 \in I$ ,  $y_0 > 0$ ,

$$(2.10) \quad T(y_0) = 2 \int_0^{\pi/2} \left( (u^{-1})'(y_0 \sin s) + (u^{-1})'(-y_0 \sin s) \right) ds.$$

So

$$(2.11) \quad \lim_{y_0 \rightarrow 0+} T(y_0) = 2\pi (u^{-1})'(0) = 2\pi / \sqrt{V''(0)}.$$

All orbits have the same period if and only if  $T(y_0)$  equals this limit value for all  $y_0 \in I$ ,  $y_0 > 0$ . By (2.9) this is equivalent to the following condition for all  $z \in I$ ,  $z > 0$ ,

$$(2.12) \quad \frac{2\pi}{\sqrt{V''(0)}} = 2 \int_0^z \frac{((u^{-1})'(r) + (u^{-1})'(-r)) dr}{\sqrt{z^2 - r^2}}.$$

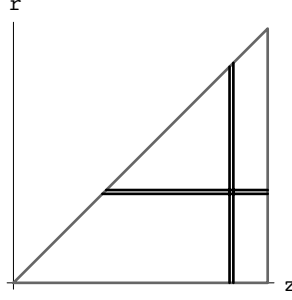


FIGURE 1.

We multiply both sides by  $z/\sqrt{y^2 - z^2}$  and we integrate from 0 to  $y \in I, y > 0$ ,

$$(2.13) \quad \begin{aligned} & \frac{\pi}{\sqrt{V''(0)}} \int_0^y \frac{z dz}{\sqrt{y^2 - z^2}} = \\ & = \int_0^y \frac{z dz}{\sqrt{y^2 - z^2}} \int_0^z \frac{((u^{-1})'(r) + (u^{-1})'(-r))}{\sqrt{z^2 - r^2}} dr. \end{aligned}$$

By the change of the integration order, see Figure 1, we get

$$(2.14) \quad \begin{aligned} & \frac{\pi y}{\sqrt{V''(0)}} = \\ & = \int_0^y \underbrace{\int_r^y \frac{z}{\sqrt{y^2 - z^2} \sqrt{z^2 - r^2}} dz}_{=\pi/2} ((u^{-1})'(r) + (u^{-1})'(-r)) dr \end{aligned}$$

$$(2.15) \quad \frac{2y}{\sqrt{V''(0)}} = u^{-1}(y) - u^{-1}(-y), \quad y \in I, y > 0.$$

We deduced this condition from (2.12), and now we see at once that it implies (2.12). By plugging  $y = u(x)$  into (2.15) we have

$$(2.16) \quad \frac{2u(x)}{\sqrt{V''(0)}} = x - h(x), \quad x \in J, x > 0.$$

This condition is equivalent to (2.7). □

Formula (2.7) corresponds to formula (6.2) in [6], the proof is included in the proof of Proposition 1 in [6] as a particular case. In [6], Section 6, there is the construction we are going to show and also another one.

From the theorem we easily get the following result.

**Corollary 2.2** (Constructing isochronous centers). *Let  $h : J \rightarrow J$  be a  $C^1$  function on an open interval  $J \subseteq \mathbb{R}$  containing 0 which satisfies*

the conditions in (2.6). Let  $\omega > 0$  and define

$$(2.17) \quad V(x) = \frac{\omega^2}{8} (x - h(x))^2, \quad x \in J.$$

Then there exists  $V''(0) = \omega^2$  and all orbits of  $\ddot{x} = -V'(x)$  which intersect the  $J$  interval of the  $x$ -axis in the  $x, \dot{x}$ -plane, are periodic and have the same period  $2\pi/\omega$ . To get  $h$  as above, we can just consider an arbitrary even  $C^1$  function on a (symmetric) open interval which vanishes at 0, then a  $\pi/4$  clockwise rotation of its graph gives the curve  $y = h(x)$  provided the interval is small enough to get a graph.

For instance, starting from  $x \mapsto x^2/\sqrt{2}$  we can (even explicitly) calculate

$$(2.18) \quad h(x) = 1 + x - \sqrt{1 + 4x}, \quad V(x) = \frac{\omega^2}{8} \left( -1 + \sqrt{1 + 4x} \right)^2.$$

Notice that the original quadratic function must be restricted to a suitable interval to get a function after rotation. Another construction of the potential energies of the isochronous centers is given in [6] (Section 6 from formula (6.3)). Finally, a simple example is also

$$(2.19) \quad h(x) = -\frac{x}{1+x}, \quad V(x) = \frac{\omega^2}{8} x^2 \left( \frac{2+x}{1+x} \right)^2.$$

Now, let us see some necessary conditions to have constant period which can be used to check whether a given function  $V$ , or its derivative  $g$ , locally gives an isochronous center or not. In the sequel  $V$  has as many derivatives as necessary. We saw how  $h$  is related to an even function. The even functions have vanishing odd derivatives at 0, so it will not be a surprise to see that the derivatives of the involution  $h$  are not arbitrary. Consider the relation  $h(h(x)) = x$ , perform 2 derivations and calculate at 0 taking into account  $h(0) = 0$  and  $h'(0) = -1$ . In this way we get an identity which shows that  $h''(0)$  can take any value. By 3 derivations and calculating at 0 we get  $h^{(3)}(0) = -3h''(0)^2/2$ . Going further we see that the even derivatives are free while the odd ones are uniquely determined by the preceding derivatives. The first 5 terms in Taylor's formula are given by

$$(2.20) \quad h(x) = -x + a x^2 - a^2 x^3 + b x^4 + (2a^4 - 3ab) x^5 + o(x^5)$$

where  $a, b$  are free parameters. Next, formula (2.7) shows that the derivatives of the isochronous  $V$  at 0 are constrained to obey some conditions. By means of (2.20) we can calculate  $V^{(4)}(0)$  and  $V^{(6)}(0)$  by the lower order derivatives and get 2 necessary conditions. Of course we can go forward to infinite conditions.

**Corollary 2.3** (Necessary conditions). *Let  $V$  admit  $V^{(6)}(0)$  and satisfy  $V(0) = V'(0) = 0$ ,  $V''(0) > 0$ . Moreover, let the origin be an*

isochronous center for  $\ddot{x} = -V'(x)$ . Then

$$(2.21) \quad V^{(4)}(0) = \frac{5V^{(3)}(0)^2}{3V''(0)}, \quad V^{(6)}(0) = \frac{7V^{(3)}(0)V^{(5)}(0)}{V''(0)} - \frac{140V^{(3)}(0)^4}{9V''(0)^3}.$$

To illustrate the necessary conditions let us use it to prove the well known lack of isochronicity of the simple pendulum  $\ddot{x} + \sin x = 0$

$$(2.22) \quad V^{(4)}(0) = \sin^{(3)}(0) = -1 \neq 0 = \frac{5\sin^{(2)}(0)^2}{3\sin'(0)} = \frac{5V^{(3)}(0)^2}{3V''(0)}.$$

Another example is

$$(2.23) \quad V(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{3}x^3 + \frac{\gamma}{4}x^4 \quad (\alpha > 0)$$

for which the second condition in (2.21) implies  $\beta = 0$ , and the first condition with  $\beta = 0$  gives  $\gamma = 0$  too.

We could also prove the above formulas in (2.21) using the approach of Barone, Cesar and Gorni [2] where the first 2 derivatives of the period function  $\mathcal{T}(x_0)$  at 0 are computed with a procedure which carries on to higher-order derivatives. With some regularity on  $V$  they find the following formulas for  $V''(0) = 1$

$$(2.24) \quad \mathcal{T}'(0) = 0, \quad \mathcal{T}''(0) = \frac{\pi}{4} \left( \frac{5}{3}V^{(3)}(0)^2 - V^{(4)}(0) \right).$$

### 3. THE DYNAMICS IN THE LAGRANGIAN FRAMEWORK

In this section we deal with the following 4-dimensional system defined by the Lagrangian function  $L$

$$(3.1) \quad \begin{aligned} L(x, y, \dot{x}, \dot{y}) &= \dot{x}\dot{y} - g(x)y, \\ \ddot{x} &= -g(x), \quad \ddot{y} = -g'(x)y, \quad g(0) = 0, \quad g'(0) > 0, \end{aligned}$$

where  $g \in C^1$  near 0 in  $\mathbb{R}$ . This system of differential equations has two first integrals  $G(x, \dot{x})$  and  $F(x, y, \dot{x}, \dot{y})$

$$(3.2) \quad \begin{aligned} G(x, \dot{x}) &= \frac{\dot{x}^2}{2} + V(x), \quad V(x) = \int_0^x g(s) ds, \\ F(x, y, \dot{x}, \dot{y}) &= \dot{y}\dot{x} + g(x)y. \end{aligned}$$

The first differential equation (3.1) separates and its dynamics was studied in the previous section. We restrict our attention to an open interval  $J$  as in Theorem 2.1 and to the orbits in the  $x, \dot{x}$ -plane which intersect  $J$ ; their union is an open neighbourhood  $C$  of  $(0, 0)$ . Let us fix  $x_0 \in J$ ,  $x_0 > 0$ , and denote by  $t \mapsto X(t, x_0)$  the periodic solution of  $\ddot{x} = -g(x)$  with  $(x_0, 0)$  as initial condition at time 0. Next, we plug  $X(t, x_0)$  into the second differential equation in (3.1) and get the linear equation with periodic coefficient, Hill's equation,

$$(3.3) \quad \ddot{y} = -g'(X(t, x_0)) y.$$

The partial derivatives of  $X(t, x_0)$  give 2 independent solutions of (3.3) as one see by derivation of the first equation for  $X(t, x_0)$ , in particular

$$(3.4) \quad \begin{aligned} \frac{\partial^2}{\partial t^2} \frac{\partial X}{\partial x_0}(t, x_0) &= \frac{\partial}{\partial x_0} \frac{\partial^2 X}{\partial t^2}(t, x_0) = \\ &= \frac{\partial}{\partial x_0} (-g(X(t, x_0))) = -g'(X(t, x_0)) \frac{\partial X}{\partial x_0}(t, x_0). \end{aligned}$$

Let us define

$$(3.5) \quad \phi(t) = \frac{\partial X}{\partial x_0}(t, x_0), \quad \psi(t) = -\frac{1}{g(x_0)} \frac{\partial X}{\partial t}(t, x_0),$$

so that

$$(3.6) \quad \begin{aligned} \phi(0) &= 1, \quad \dot{\phi}(0) = 0, \quad \psi(0) = 0, \quad \dot{\psi}(0) = 1, \\ \dot{\psi}(t) \phi(t) - \dot{\phi}(t) \psi(t) &= 1. \end{aligned}$$

Let us denote by  $\mathcal{T}(x_0)$ , briefly  $\tau$ , the period of  $t \mapsto X(t, x_0)$ , then  $t \mapsto \phi(t + \tau)$  is also a solution of (3.3), so a linear combination of  $\phi$  and  $\psi$ , and the previous values at 0 give

$$(3.7) \quad \phi(t + \tau) = \phi(\tau) \phi(t) + \dot{\phi}(\tau) \psi(t) = \phi(t) + \dot{\phi}(\tau) \psi(t)$$

where we used  $\phi(\tau) = 1$  which comes from the last equality in (3.6) with  $t = \tau$  if we notice that  $\psi(\tau) = 0$  and  $\dot{\psi}(\tau) = 1$ . Taking the derivative, calculating at  $t = \tau$ , and taking into account  $\dot{\psi}(n\tau) = 1$ , we have

$$(3.8) \quad \dot{\phi}(t + \tau) = \dot{\phi}(t) + \dot{\phi}(\tau) \dot{\psi}(t) \implies \dot{\phi}((n+1)\tau) = \dot{\phi}(n\tau) + \dot{\phi}(\tau),$$

for all  $n \in \mathbb{Z}$ . This last result is equivalent to

$$(3.9) \quad \dot{\phi}(n\tau) = n\dot{\phi}(\tau), \quad \forall n \in \mathbb{Z}.$$

If  $\dot{\phi}(\tau) = 0$  the solution  $\phi$  is periodic, otherwise it is unbounded. A necessary and sufficient condition to have the former case is  $\dot{\phi}(\tau) = 0$  namely

$$(3.10) \quad \frac{\partial^2 X}{\partial t \partial x_0}(\mathcal{T}(x_0), x_0) = 0.$$

To go ahead we need the differentiability of the period function

**Proposition 3.1.** *The period  $\mathcal{T}(x_0)$  of  $t \mapsto X(t, x_0)$  is a  $C^1$  function on  $\{x_0 \in J : x_0 > 0\}$ .*

*Proof.* By formula (2.10), for  $x_0 \in J$ ,  $x_0 > 0$ , we have

$$(3.11) \quad \begin{aligned} \mathcal{T}(x_0) &= \\ &= 2 \int_0^{\pi/2} \left( (u^{-1})'(u(x_0) \sin s) + (u^{-1})'(-u(x_0) \sin s) \right) ds. \end{aligned}$$

The function  $u$  is a  $C^1$  diffeomorphism as in the previous section. In the stronger hypothesis of this section,  $g \in C^1$ , we have that  $u(x) = \sqrt{V(x)}$



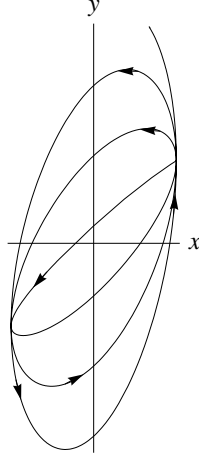


FIGURE 2.

for  $x > 0$  is  $C^2$  as well as  $V = g'$  and its inverse  $u^{-1}(y)$  for  $y > 0$ . This proves the result.  $\square$

Now, we can differentiate  $\partial_t X(\mathcal{T}(x_0), x_0) = 0$  with respect to  $x_0$

$$(3.12) \quad \frac{\partial^2 X}{\partial t^2}(\mathcal{T}(x_0), x_0) \mathcal{T}'(x_0) + \frac{\partial^2 X}{\partial x_0 \partial t}(\mathcal{T}(x_0), x_0) = 0.$$

We just remind that  $(\partial^2 X / \partial t^2)(\mathcal{T}(x_0), x_0) = -g(x_0) < 0$  for  $x_0 > 0$ , and have that condition (3.10) is equivalent to

$$(3.13) \quad \mathcal{T}'(x_0) = 0.$$

This is the condition in order  $\phi$  to be periodic. Since the linear combination of  $\phi$  and  $\psi$  gives the general solution, we have just proved the following

**Proposition 3.2.** *Let us fix  $x_0 \in J$  with  $x_0 > 0$ , then the solutions to the  $\mathcal{T}(x_0)$ -periodic equation (3.2) are either all  $\mathcal{T}(x_0)$ -periodic, or all unbounded but those proportional to  $t \mapsto \partial_t X(t, x_0)$ . A necessary and sufficient condition for the former case is (3.13).*

In Figure 2 the curve  $(x(t), y(t)) = (X(t, x_0), \phi(t))$ , namely the solution to system of differential equations (3.1) with  $x(0) = x_0 > 0$ ,  $y(0) = 1$ ,  $\dot{x}(0) = \dot{y}(0) = 0$ , in a non-periodic case. The initial point  $(x_0, 1)$  and the point  $(h(x_0), g(x_0)/g(h(x_0)))$  are crossed at any period.

Now, notice that the origin in  $\mathbb{R}^4$  is an unstable equilibrium for (3.1) if and only if we can find  $x_0 > 0$  arbitrarily close to 0 such that equation (3.2) has unbounded solutions. So by Proposition 3.2 we have

**Theorem 3.3** (Stability and weak instability). *Let  $g$  be a  $C^1$  function near 0 in  $\mathbb{R}$  with  $g(0) = 0$ ,  $g'(0) > 0$ , then the origin in  $\mathbb{R}^4$  is a stable equilibrium for the system (3.1), if and only if the origin in  $\mathbb{R}^2$  is a locally isochronous center for  $\ddot{x} = -g(x)$ , in this case all orbits of*

(3.1) with  $(x, \dot{x})$  near  $(0, 0)$  are periodic and have the same period. If the equilibrium is unstable, then there is a sequence of initial data which converges to the origin whose corresponding solutions are unbounded. The instability is weak, by this we mean that there are no asymptotic motions to the equilibrium, namely no motions which tends to the origin as  $t \rightarrow -\infty$ .

We already proved the first part of the statement, for the last, let us only remark that the distance of an orbit from the origin in  $\mathbb{R}^4$  is greater than the distance of the projection in the  $x, \dot{x}$ -plane which is strictly positive unless the projection is  $(0, 0)$  and in this case the second differential equation in (3.1) gives the harmonic oscillator  $\ddot{y} = -g'(0)y$ . An explicit function  $g$  to give such an example is  $\sin(x)$  (see (2.22)). However, as we know from Section 2, almost all functions  $g$  as in the statement of the theorem, give instability.

Examples of the rare functions which give stable equilibria are (see (2.18) and (2.19))

$$(3.14) \quad g(x) = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1+4x}} \right), \quad g'(0) = 1,$$

$$(3.15) \quad g(x) = \frac{1}{4} \left( 1 + x - \frac{1}{(1+x)^3} \right), \quad g'(0) = 1.$$

For all these functions which give stable equilibria, at least whenever  $g \in C^2$  the system (3.1) admits a (positive definite) Lyapunov function which is a further first integral; this additional first integral is smooth in a neighbourhood of the origin in  $\mathbb{R}^4$  and at least continuous at the origin as proved by Barone and Cesar in [3]. In the following statement  $C \subseteq \mathbb{R}^2$  is the set defined below formula (3.2).

**Theorem 3.4** (Lyapunov functions). *Let  $g : J \rightarrow \mathbb{R}$  be a  $C^2$  function on the open interval  $J \subseteq \mathbb{R}$  with  $0 \in J$ ,  $g(0) = 0$ ,  $g'(0) > 0$ . Suppose that all orbits of  $\ddot{x} = -g(x)$  which intersect the  $J$  interval of the  $x$ -axis in the  $x, \dot{x}$ -plane, are periodic and have the same period  $2\pi/\sqrt{g'(0)}$ , so the origin in  $\mathbb{R}^4$  is a stable equilibrium for the system (3.1). Then there exists*

$$(3.16) \quad E(x, y, \dot{x}, \dot{y}) = a(x, \dot{x})\dot{y}^2 + b(x, \dot{x})y\dot{y} + c(x, \dot{x})y^2, \quad a, b, c : C \rightarrow \mathbb{R},$$

*continuous and positive definite function, which is a (global) first integral for the system (3.1).*

The first integral is obtained by means of the functions in (3.5) and a suitable inverse function, see [3] for details.

Let us also notice that

**Proposition 3.5** (Eigenvalues). *Let  $g$  be a  $C^1$  function near 0 in  $\mathbb{R}$  with  $g(0) = 0$ ,  $g'(0) > 0$ , then the linearization of the system (3.1) at the origin in  $\mathbb{R}^4$  has the double eigenvalues  $\pm i\sqrt{g'(0)}$ .*

Let us remark that we can easily construct unstable cases with some  $x_0$  at which (3.13) holds and so the corresponding orbits are all periodic. If we want the period function composed with  $u^{-1}$  to be for instance  $T(y_0) = 2\pi(1 - y_0^2 + y_0^4)$ , whose derivative vanishes at  $y_0 = 1/\sqrt{2}$  we integrate

$$(3.17) \quad \int_0^{y_0} \frac{zT(z)}{\sqrt{y_0^2 - z^2}} dz = 2\pi \left( y_0 - \frac{2}{3}y_0^3 + \frac{8}{15}y_0^5 \right).$$

So we get the relation

$$(3.18) \quad u^{-1}(y_0) - u^{-1}(-y_0) = 2 \left( y_0 - \frac{2}{3}y_0^3 + \frac{8}{15}y_0^5 \right).$$

The symmetric choice  $u^{-1}(-y_0) = -u^{-1}(y_0)$  gives

$$(3.19) \quad u^{-1}(y_0) = y_0 - \frac{2}{3}y_0^3 + \frac{8}{15}y_0^5.$$

By inversion we get  $u(x)$ , then  $V(x) = u(x)^2/2$  and finally  $g(x) = V'(x)$ . There is an isolate  $x_0$  at which (3.13) holds. We can also imagine more complicate examples where these  $x_0$  accumulate at 0.

Finally, let us mention that most of the results in Section 3 were found in [7] by a different (more complicated) proof which included other differential equations in a family for which the present system was a particular case. However, in [7] I had not realized the Lagrangian character of the present case which is very important to our purposes and will be exploited in the next section.

#### 4. COMPLETE INTEGRABILITY

In this section we write  $q = (q_1, q_2) = (x, y)$ . The Legendre transformation takes the Lagrangian system (3.1) into

$$(4.1) \quad \begin{aligned} H(q, p) &= p_1 p_2 + g(q_1) q_2, \\ \dot{q}_1 &= \frac{\partial H}{\partial p_1}(q, p) = p_2, & \dot{q}_2 &= \frac{\partial H}{\partial p_2}(q, p) = p_1, \\ \dot{p}_1 &= -\frac{\partial H}{\partial q_1}(q, p) = -g'(q_1) q_2, & \dot{p}_2 &= -\frac{\partial H}{\partial q_2}(q, p) = -g(q_1). \end{aligned}$$

The Hamiltonian function corresponds to the first integral  $F$  while the first integral  $G$  becomes

$$(4.2) \quad K(q, p) = \frac{p_2^2}{2} + V(q_1), \quad V(q_1) = \int_0^{q_1} g(s) ds.$$

Our phase space is

$$(4.3) \quad M = \{(q_1, q_2, p_1, p_2) \in \mathbb{R}^4 : (q_1, p_2) \in C, (q_2, p_1) \in \mathbb{R}^2\}$$

where  $C \subseteq \mathbb{R}^2$  is an open set mentioned in Section 3 between formulas (3.2) and (3.3) so to have a global *center* in the  $q_1, p_2$ -plane. The

Hamiltonian vector fields

$$(4.4) \quad \Omega \nabla H(q, p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{q_1} H(q, p) \\ \partial_{q_2} H(q, p) \\ \partial_{p_1} H(q, p) \\ \partial_{p_2} H(q, p) \end{pmatrix} = \begin{pmatrix} p_2 \\ p_1 \\ -g'(q_1)q_2 \\ -g(q_1) \end{pmatrix}$$

$$(4.5) \quad \Omega \nabla K(q, p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{q_1} K(q, p) \\ \partial_{q_2} K(q, p) \\ \partial_{p_1} K(q, p) \\ \partial_{p_2} K(q, p) \end{pmatrix} = \begin{pmatrix} 0 \\ p_2 \\ -g(q_1) \\ 0 \end{pmatrix}$$

are both *complete* on  $M$ , namely their integral curves are all defined on the whole  $\mathbb{R}$ . Indeed, this is clear for (4.4) since we have a global center on the  $q_1, p_2$ -plane and linear equations in  $q_2, p_1$ ; for (4.5) we simply integrate and remind (4.3)

$$(4.6) \quad \begin{aligned} q_1(t) &= q_1(0), & q_2(t) &= q_2(0) + p_2(0)t, \\ p_1(t) &= p_1(0) - g(q_1(0))t, & p_2(t) &= p_2(0). \end{aligned}$$

The functions  $H, K$  are *in involution*, indeed their Poisson brackets vanish:

$$(4.7) \quad \begin{aligned} \{H, K\} &= \frac{\partial H}{\partial q} \cdot \frac{\partial K}{\partial p} - \frac{\partial K}{\partial q} \cdot \frac{\partial H}{\partial p} = \\ &= \begin{pmatrix} g'(q_1)q_2 \\ g(q_1) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p_2 \end{pmatrix} - \begin{pmatrix} g(q_1) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} p_2 \\ p_1 \end{pmatrix} = 0. \end{aligned}$$

The vectors  $\nabla H(q, p), \nabla K(q, p)$  are *linearly independent* at each point of the set

$$(4.8) \quad N := \{(q, p) \in M : (q_1, p_2) \neq (0, 0)\}$$

which is *invariant* for both the Hamiltonian vector fields  $\Omega \nabla H$  and  $\Omega \nabla K$ . Indeed, the condition  $\alpha \nabla H(q, p) + \beta \nabla K(q, p) = 0$  implies  $\alpha g(q_1) = 0$  and  $\alpha p_2 = 0$ , so  $\alpha = 0$  since  $(q_1, p_2) \neq (0, 0)$ ; then it also implies  $\beta g(q_1) = 0$  and  $\beta p_2 = 0$  so  $\beta = 0$ .

Let  $\Gamma \subset N$  be a nonempty component of a level set of  $(H, K)$ , then we see at once that it is not compact since it contains the unbounded curve  $p_1 \mapsto (q_1, c/g(q_1), p_1, 0)$  where  $c$  is the value of  $H$  and  $V(q_1) \neq 0$  is the one of  $K$ . By a well known theorem (see Theorem 3, Chapter 4, in Arnold, Kozlov and Neishtadt [1]),  $\Gamma$  is diffeomorphic to  $\mathbb{S}^1 \times \mathbb{R}$ . More precisely

**Theorem 4.1** (Complete integrability). *Let  $g$  be a  $C^1$  function near 0 in  $\mathbb{R}$  with  $g(0) = 0$ ,  $g'(0) > 0$ . The functions  $H, K : M \rightarrow \mathbb{R}$ , in (4.1) and (4.2), are in involution on  $M$ ,  $\dim M = 4$ , and the Hamiltonian vector fields  $\Omega \nabla H$  and  $\Omega \nabla K$  are complete on  $M$ . The set  $N$  in (4.8) is invariant for  $\Omega \nabla H$  and  $\Omega \nabla K$  and  $H, K$  are independent on  $N$ . Let  $\Gamma \subset N$  be a nonempty component of a level set of  $(H, K)$  then it is diffeomorphic to  $\mathbb{S}^1 \times \mathbb{R}$  and there are coordinates  $\phi \bmod 2\pi$  and  $z$  on*

$\mathbb{S}^1 \times \mathbb{R}$  such that the differential equations defined by the vector field  $\Omega \nabla H$  on  $\Gamma$  take the form

$$(4.9) \quad \dot{\phi} = \omega, \quad \dot{z} = v \quad (\omega, v = \text{const}).$$

If  $(x_0, 0)$  belongs to the projection of  $\Gamma$  on the  $q_1, p_2$ -plane and  $\mathcal{T}(x_0)$  is the period of the first component of any integral curve of  $\Omega \nabla H$  on  $\Gamma$ , then  $\omega = 2\pi/\mathcal{T}(x_0)$  and  $v = 0$  if and only if  $\mathcal{T}'(x_0) = 0$ .

We already proved everything in the statement of the theorem but the last sentence which is a straight consequence of  $\mathcal{T}'(x_0) = 0$  being the necessary and sufficient condition for the solutions on  $\Gamma$  to be all periodic (see Proposition 3.2), and (4.9) gives periodic solutions if and only if  $v = 0$ .

For the *isochronous* systems, namely whenever the period function is constant and all integral curves of  $\Omega \nabla H$  in  $M$  have the same period, we know from Theorem 3.4 that a further first integral exist which is quadratic in  $q_2, p_1$ . Of course we do not expect this first integral to be in involution with  $K$ . In the next section we investigate the (isochronous) cases where a third first integral quadratic in the momenta  $p_1, p_2$  exists and we arrive at explicit formulas for it.

## 5. EXPLICIT SUPERINTEGRABLE SYSTEMS

Let  $g$  be a  $C^1$  function near 0 in  $\mathbb{R}$  with  $g(0) = 0, g'(0) > 0$ . Suppose that the system (4.1) has a first integral  $W$  of the form

$$(5.1) \quad A(q_1, q_2)p_1^2 + B(q_1, q_2)p_1p_2 + C(q_1, q_2)p_2^2 + U(q_1, q_2).$$

Then the equation  $\{H, W\} = 0$  give a cubic polynomial in  $p_1, p_2$ . The coefficients vanish if and only if

$$(5.2) \quad \partial_2 A = 0, \quad \partial_1 C = 0, \quad \partial_1 A + \partial_2 B = 0, \quad \partial_1 B + \partial_2 C = 0,$$

$$(5.3) \quad \begin{aligned} \partial_1 U(q_1, q_2) &= 2C(q_1, q_2)g(q_1) + B(q_1, q_2)g'(q_1)q_2, \\ \partial_2 U(q_1, q_2) &= 2A(q_1, q_2)g'(q_1)q_2 + B(q_1, q_2)g(q_1). \end{aligned}$$

The conditions in (5.2) are the same as those in (3.2.2) in Hietarinta [4] where the celebrated Darboux problem is treated (only the order of the coordinates is changed since  $\dot{q}_1 = p_2, \dot{q}_2 = p_1$ ). They have the following solution where  $a, b_1, b_2, c_1, c_2, c_3 \in \mathbb{R}$  are arbitrary

$$(5.4) \quad \begin{aligned} A(q_1, q_2) &= aq_1^2 + b_1q_1 + c_1 \\ B(q_1, q_2) &= -2aq_1q_2 - b_1q_2 - b_2q_1 + c_3 \\ C(q_1, q_2) &= aq_2^2 + b_2q_2 + c_2 \end{aligned}$$

Plugging (5.4) in (5.3) we get the following condition for the integrability of  $U$

$$(5.5) \quad \begin{aligned} 3(b_2 + 2aq_2)g(q_1) &= \\ &= 3(b_1 + 2aq_1)q_2g'(q_1) + 2(c_1 + q_1(b_1 + aq_1))q_2g''(q_1). \end{aligned}$$

For  $q_2 = 0$  we get  $3b_2g(q_1) = 0$  which implies  $b_2 = 0$ , and back to (5.5) we have

$$(5.6) \quad 6ag(q_1) = 3(b_1 + 2aq_1)g'(q_1) + 2(c_1 + q_1(b_1 + aq_1))g''(q_1).$$

For  $a = 0$ ,  $g(0) = 0$  and  $g'(0) = \omega^2$ , this equation gives

$$(5.7) \quad g(q_1) = \frac{\omega^2}{\lambda} \left( 1 - \frac{1}{\sqrt{1 + 2\lambda q_1}} \right)$$

where  $\lambda = b_1/(2c_1)$  (for  $c_1 = 0$  there are no solutions). The values  $\lambda = 2$  and  $\omega = 1$  give (3.14). The first integral (5.1) for this  $g$  is easily obtained with  $b_1 = 2\lambda c_1$ ,  $a = 0$ ,  $b_2 = 0$

$$(5.8) \quad c_1 p_1^2(1 + 2\lambda q_1) + p_1 p_2(c_3 - 2\lambda c_1 q_2) + c_2 p_2^2 + \\ + c_2 \frac{\omega^2}{\lambda^2} \left( -1 + \sqrt{1 + 2\lambda q_1} \right)^2 + c_1 \omega^2 q_2^2 + q_2(c_3 - 2\lambda c_1 q_2)g(q_1) + d$$

where  $d \in \mathbb{R}$ . For  $c_1 = c_3 = d = 0$ ,  $c_2 = 1/2$ , we have the first integral  $K$  we already know, while  $H$  corresponds to the choice  $c_1 = c_2 = d = 0$ ,  $c_3 = 1$ . For  $c_1 = c_2 = 1$ ,  $c_3 = d = 0$  we have the first integral

$$(5.9) \quad p_1^2(1 + 2\lambda q_1) - 2\lambda q_2 p_1 p_2 + p_2^2 + 2V(q_1) + (\omega^2 - 2\lambda g(q_1))q_2^2.$$

This function is positive definite near the origin of  $\mathbb{R}^4$  as we check at once. This is why we have Lyapunov stability of the equilibrium and compact orbits.

We dealt with the particular case  $a = 0$ . In the sequel  $a \neq 0$  and we fix  $a = 1$  dividing the first integral by  $a$ . The coefficient of  $g''$  in equation (5.6) is a quadratic function in  $q_1$  with discriminant  $b_1^2 - 4c_1$ . Let us consider first the case where this vanishes, namely  $c_1 = b_1^2/4$ . If  $b_1 = 0$  then (5.6) with  $g(0) = 0$ ,  $g'(0) = \omega^2$ , gives the trivial function  $g(x) = \omega^2 x$ , while for  $\lambda := 2b_1 \neq 0$  we get the following solution with  $g(0) = 0$ ,  $g'(0) = \omega^2$ :

$$(5.10) \quad g(q_1) = \frac{\omega^2}{4} \left( \lambda + q_1 - \frac{\lambda^4}{(\lambda + q_1)^3} \right)$$

Notice that this formula includes (3.15) for  $\omega = 1$  and  $\lambda = 1$ . The first integral (5.1), which corresponds to (5.10), with  $c_3 = 0$ ,  $c_2 = 1$ , and which vanishes at the origin of  $\mathbb{R}^4$ , is

$$(5.11) \quad p_1^2(\lambda + q_1)^2 - 2p_1 p_2(\lambda + q_1)q_2 + p_2^2(1 + q_2^2) + \\ + \frac{\omega^2}{4} \left( (\lambda + q_1)^2 + \lambda^4 \frac{1 + 4q_2^2}{(\lambda + q_1)^2} \right) - \frac{\lambda^2 \omega^2}{2}.$$

We can check that at the origin of  $\mathbb{R}^4$  its gradient vanishes and the Hessian matrix is the diagonal matrix  $2(\omega^2, \lambda^2 \omega^2, \lambda^2, 1)$ , so we have again a (positive definite near the origin) Lyapunov function for all values of the parameter  $\lambda \neq 0$ .

Now we are ready to deal with the generic case. Let  $a = 1$  and  $b_1^2 - 4c_1 \neq 0$ , we also consider  $b_1 \neq 0$  and  $c_1 \neq 0$  since  $b_1 = 0$  gives the trivial solution and  $c_1 = 0$  no solutions at all. Then the solution of the equation (5.6) with  $g(0) = 0$  and  $g'(0) = \omega^2 > 0$  is

$$(5.12) \quad g(q_1) = \frac{2c_1\omega^2}{(b_1^2 - 4c_1)^2} \left( (b_1^2 + 4c_1)(b_1 + 2q_1) + \right. \\ \left. + b_1 \frac{b_1^2 - 4c_1 - 2(b_1 + 2q_1)^2}{\sqrt{1 + q_1(b_1 + q_1)/c_1}} \right).$$

The first integral (5.1) with  $c_3 = 0$ ,  $c_2 = c_1$ , divided by  $c_1$ , and which vanishes at the origin of  $\mathbb{R}^4$ , is

$$(5.13) \quad p_1^2 + p_2^2 + \frac{1}{c_1} (p_1 q_1 - p_2 q_2) (p_1(b_1 + q_1) - p_2 q_2) + \\ + \frac{\omega^2}{(b_1^2 - 4c_1)^2} \left( 8b_1^2 c_1^2 + 4c_1(b_1^2 + 4c_1)(b_1 + q_1)q_1 + (16c_1^2 - b_1^4)q_2^2 \right) + \\ + \frac{2b_1\omega^2}{(b_1^2 - 4c_1)^2} \frac{(b_1 + 2q_1) \left( -4c_1(c_1 + (b_1 + q_1)q_1) + (b_1^2 - 4c_1)q_2^2 \right)}{\sqrt{1 + q_1(b_1 + q_1)/c_1}}.$$

We can check that at the origin of  $\mathbb{R}^4$  its gradient vanishes and the Hessian matrix is the diagonal matrix  $2(\omega^2, \omega^2, 1, 1)$ , so we have a positive definite function near the origin.

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